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FLEXURE AND TRANSVERSE RESISTANCE OF BEAMS.

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§ 1. It is well known that materials like cast iron, with an ultimate tensile resistance much less than their ultimate resistance to crushing, show, under transverse strain, reduced to longitudinal strain by the usual formula, much greater tensile strength than when the strains are actually applied in a longitudinal direction.

§ 2. The subject has excited considerable interest for years. It was discussed by this Society in connection with the paper of Mr. L. Nickerson, giving "*A record of experiments showing the character and position of Neutral Axes as seen by polarized light*," followed by that of Gen. J. D. Barnard on "*Resistance of Beams to Flexure*,"* and articles in relation thereto containing various theories and explanations, have frequently appeared in the technical journals. Cast iron is not now much used in construction, but it is known that there are also great discrepancies in the application of the theory of transverse strain to other materials, particularly particularly wrought iron and steel, so that a re-examination of the subject is considered of practical value and interest.

* LXXXIII., Trans. Am. Soc. Civil Engrs., Vol. III., p. 31, and Paper XCI., p. 123, same Vol.

§ 3. Mr. B. B. Stoney, in his excellent work on "The Theory of Strains," § 130, refers to the subject as follows :

"Mr. Hodgkinson endeavors to explain this discrepancy by a change in the position of the neutral axis as soon as the limit of elastic reaction of the horizontal fibres has been passed, and gives some reasons for this hypothesis in his "*Experimental Researches on the Strength of Cast Iron*," p. 384. This seems a plausible hypothesis, for if the neutral axis of a solid rectangular cast iron girder approach its compressed edge as the weight increases, and after the limit of tensile elasticity has been passed by the fibres along the extended edge, we shall have a larger proportion than one-half the girder subject to tension, and consequently the total horizontal tensile strain may exceed that derived from our theory, which assumes that the neutral axis always passes through the center of gravity of the cross section (68). Mr. Hodgkinson concludes from his experiments that the neutral axis of a retangular girder divides the depth in the proportion of $\frac{1}{5}$ or $\frac{1}{6}$ at the time of fracture—that is, that the compressed section is to the extended section in the inverse proportion of the compressive to the tensile strength of the material."

§ 4. After stating that this view is corroborated by experiments of Duhamel and the elder Barlow, but controverted by Mr. W. H. Barlow, and apparently disproved by his micrometric experiments, as well as the experiments of Sir D. Brewster, with polarized light, Mr. Stoney concludes :

"The whole question, it must be confessed, is one of great difficulty, and may require numerous experiments before it can be satisfactorily solved. One practical inference is, however, of great importance, namely, that *the tearing and crushing strengths of materials derived from experiments on the transverse strength of solid girders are often erroneous*, and have even led astray men of such capacity as Tredgold."

§ 5. The experiments of Mr. W. H. Barlow, referred to, were made with beams $7\frac{1}{2}$ feet long, 6 inches deep, and 2 inches thick, on the sides of which were cast small vertical ribs, spaced one foot apart, and provided with small holes to receive pins on a micrometric apparatus, used to show the elongations and compressions at 9 different positions in the depth of the beam. The published results for the space between the central ribs, though not agreeing precisely among themselves, show clearly that in that case the neutral axis did not shift materially for loads about $\frac{3}{4}$ of those required to break the beam.*

* See papers of Mr. W. H. Barlow, Phil. Trans., 1855, p. 225 ; Ditto, 1857, p. 463.

Mr. W. H. Barlow advanced a theory based on these experiments, that since the horizontal lamina of a beam are attached together by the force of cohesion, there arises a resistance to bending the mass which is in addition to that developed in elongating the fibres.* This theory has probably received the most attention of the many proposed, but it is evident that, at the section of a beam where the moment is a maximum, the shearing or bending strains are zero, and the fibres subjected only to horizontal strain; yet this is the section for which the maximum resistance is determined, and where the discrepancy is the greatest, the actual resistance being often as much as 2 to 2½ times that determined by accepted theories.

§ 6. Mr. Hodgkinson announced his theory in his work on "*The Strength and Properties of Cast Iron*," published in 1846, and three years after, experiments made by him on the direct tensile and compressive resistances of cast iron, with a view of determining the position of the neutral axis under transverse strain, were published in the "*Report of the Commissioners appointed to inquire into the application of Iron to Railway Structures*," of which commission he was a member.

As will be shown hereafter, the results of these experiments are not in conflict with those published by Mr. W. H. Barlow in 1855, if compared within the same limits, but show that the neutral axis must shift during the higher strains, just before rupture, though ordinarily to so slight an extent as to account for a small portion only of the discrepancy.

§ 7. Frequently verbal expressions are heard that of course the neutral axis must shift to produce equilibrium, since the ultimate compressive resistance of cast iron is some six times the tensile resistance. The discussions of the previously mentioned paper of Mr. Nickerson by Gen. Barnard in his paper, and Prof. Wood in connection therewith, indicate this general view. Evidently a higher strain in compression than in tension can be obtained only when the upper and lower sections are inversely proportional to the ultimate resistances, for the reason that the moduli of elasticity in tension and compression are, within certain limits, so nearly the same that the compressive section being connected by the forces of cohesion to the tensile section, cannot move

* This resistance has been compared to that which would be necessary to prevent a number of boards from slipping upon each other when laid together upon end supports and loaded. [See Mr. Nickerson's paper above referred to]. This is not considered a good illustration of Mr. Barlow's theory. Evidently the thinner the boards the less of such resistance would be developed; so for consecutive lamina there would be no such resistance.

through a greater angle, and, therefore, the compressed side of a symmetrical section cannot utilize its surplus strength until the outer fibres in tension actually begin to stretch rapidly preliminary to parting. This explains the value of the Hodgkinson beam section, as with it the neutral axis can shift toward the *tensile* edge to permit the compressed section to move through a sufficient angle to utilize the ultimate resistance of the material. It is evident, therefore, that a beam with a rectangular or other symmetrical section must always fail in tension, though exhibiting much greater resistance than has yet been satisfactorily explained.

§ 8. It is the purpose of the writer to present the results of experiments showing the relation of the longitudinal strains to the corresponding elongations and compressions, between the tensile and compressive limits of rupture; then to show mathematically the relation between the transverse and direct tensile and compressive resistances of the material, and finally to develop and plot the isodynamic curves, and from their relative position show the probability that there is an equalization of strain between the outer fibres and those nearer the neutral axis, so that the latter, at the section of maximum moment, offer more resistance than is due to accepted theory, and the former are correspondingly relieved; which phenomenon, in connection with a slight shifting of the neutral axis, will account for the increased resistance shown by experiment.

§ 9. In this connection it has been thought desirable to collect together in Appendix A a number of formula relating to flexure and transverse resistance, applicable to both solid and skeleton girders, and extend the same in forms convenient for reference during the present investigation, and others having a different bearing, which it is proposed to submit. In these formula the modulus or coefficient of elasticity has been, as is customary, considered uniform, within what are known as elastic limits. This is not exactly the case, but is practically so in most instances, for the same piece of the same material. It is well known that different pieces of similar material may have very different coefficients of elasticity, which fact must be provided for in some forms of construction.

§ 10. Evidently the principles affecting the flexure and rupture of materials under transverse strain depend upon the same principles; and a truly general equation should give accurate results for every

strain within or beyond the elastic limit. Representing the direct elongations of a unit of the material by x' and the corresponding strain or unit force resulting from such elongation by y', a general equation requires the previous determination of the value of $\frac{dy'}{dx'}$, or the rate of strain for different elongations which shall be correct between the tensile or positive and the compressive or negative limits of rupture.

§ 11. All the physical properties of materials are so dependent upon their chemical and molecular structure, that their relations must necessarily be determined by experiment. By noting and tabulating the elongations and corresponding strains, curves may be constructed like those shown in the plate, in which the elongations and compressions are respectively laid off as abscissa to the right and left of a vertical axis, and the corresponding strains or unit forces developed as ordinates above and below a horizontal axis, the origin being designated O, in either case. Necessarily for clearness, the scale for the elongations is much more exaggerated than that for the strains. There has been no attempt to make the scales uniform, the object being simply to plot all the results in the most convenient manner on the same cross section paper.

§ 12. Curve 1 is plotted from the results of experiments made by Mr. Hodgkinson in the years 1847–9 on the direct extension and compression of cast iron bars of the grade known as Low Moor No. 2. The bars extended were 50 feet long in connected sections of 10 feet; those compressed were 10 feet long, and secured in a frame to prevent buckling, being jarred occasionally, when under strain, to give them freedom to shorten. Curve 2 was similarly plotted from the average results of similar trials of a number of different kinds of iron, that above mentioned included. All the bars were one inch square.* Curve 3 was plotted from tests made by Capt. T. J. Rodman, U. S. Ordnance, of a specimen of gun iron marked *DO*, of 82 per cent. Greenwood and 18 per cent. Salisbury cast iron, remelted, made preliminary to casting a 15-inch gun. The specimen extended was 30 inches long—that compressed 10 inches long, both being 1.382 inches in diameter.† Curve 4 shows the results of similar tests of another sample of gun iron. Curves 5 and

* See page 59, Appendix Railway Commissioners' Report, previously mentioned.

† Rodman's Report of Experiments on Properties of Metal for Cannon and the qualities of Cannon Powder, 1861.

6 show the results of experiments made in the Stevens Institute laboratory on the properties of Nos. 2 and 4 Salisbury iron.* Curve 7 shows the results of a test made of the extension of a bar of cast iron by Mr. Wm. Kent, M. E.† Curves 8 are examples made autographically by the torsion testing machine of Prof. Thurston, of the Stevens Institute.‡ Curve 9 shows the results of experiments made by Mr. David Kirkaldy on the direct elongation and compression of Swedish Bessemer steel.§ The specimens were 100 inches long, about $2\frac{1}{4}$ inches wide, and from $\frac{1}{8}$ to $\frac{3}{8}$ inch in thickness. The compressed pieces were kept straight by Kirkaldy's trough apparatus and its adjustments. These specimens, with those used in other tests were exhibited at the Centennial Exhibition, and excited great interest.

§ 13. Curves 1, 2 and 3 show at a glance that cast iron obtains its greater ultimate strength in compression, compared with that in tension, simply by increased compression with a nearly constant rate of strain instead of by an increase in the rate of strain.

§ 14 Calling $\frac{dy'}{dx'}$, or the rate of resistance for varying elongations, the *co-efficient of resistance*, this will be proportioned to the modulus or co-efficient of elasticity within the elastic limits, and if constant within such limits as generally assumed would be represented on the diagrams by an inclined straight line for a certain distance above and below the horizontal axis. It will be seen that this is not so in all cases. Curve 3 practically shows this feature, but many others plotted from experiments with similar material do not. Generally in fact the co-efficient of resistance (and elasticity) is less for limited compression than for limited elongations, which is shown clearly by curves 1 and 2, and Capt. Rodman's experiments indicate that the same thing occurs with iron of a higher grade, the compression branch crossing the extension, when both are plotted above the horizontal axis, as was his custom. A specimen of this is shown by curve 4, in respect to which Capt. Rodman calls attention to the fact that with such material under transverse strain the

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* Plotted from separate tension and compression curves published in "Railroad Gazette," November, 1877.

† Kent on Strength of Materials, Van Nostrand's Magazine, Jan. 1878.

‡ Trans. Am. Soc. Civil Engrs., LXXVI and LXXXII, Vols. II and III.

§ Kirkaldy's Experimental Inquiry into the Mechanical Properties of Fagestra steel, 1873.

In this connection the writer desires to acknowledge his indebtedness to Acting Prof. Denton for references to valuable works in the library of the Stevens Institute, and transcripts from the records of the laboratory.

neutral axis must first approach the extended, and afterward the compressed edge.*

§ 15. A number of calculations have been made to ascertain the strain on the outer fibres of a beam, on the basis that the strains for different elongations are proportioned to the ordinates of an experimental curve, instead of directly as the elongations, as in the ordinary theory. To accomplish this it has been found most convenient to derive the relations from the curves as plotted, by finding an equation closely adapted to the particular curve shown, thereby obtaining a value for a function of x, which is substituted in the differential equation in place of one of the factors, x; and, after integration, the special values are divided out and the results expressed in terms of the hight of the section, as is usual. The compression branches are so nearly straight lines within the limits due to transverse strain, that they have been so considered; but provision is made for the angle at which such lines meet the extension curves. Portions of the latter are also considered straight lines when the plotted curves warrant. The equations used and calculations in detail are given in Appendix B, § 47 to 51. It is there shown, § 52, that, by the ordinary formulæ, with neutral axis in the center, the strain Y_0 on the outer fibre $= 6\, M \div b\, D^2$, M being the moment; b the breadth and D the total depth of the section.

§ 16. The tensile branch of Curve 1, showing the results of Mr. Hodgkinson's experiments with Low Moor cast iron No. 2, was found, as plotted, to be closely represented by the equation $y = 1.137\, x - .018\, x^2$. The tangent at the origin $\frac{dy}{dx} = 1.137$, but, being evidently less for the compressive branch, the value $g = .967$ was taken directly from the curve and used in the calculations. Evidently in a case of this kind the neutral axis must first shift toward the extended, and afterward toward the compressed edge, and the final result due to considering the ordinates of the experimental curve may not vary greatly from that given by the ordinary

* The value of the curves in showing the physical properties of materials as illustrated by those from the simple torsion apparatus of Prof. Thurston so impressed the writer that in presenting plans for a testing machine of 400 tons capacity to the Government Board in the year 1875, he embodied, as part of the design an autographic apparatus to show the relations of the strains to the direct changes of length of the specimen. The distinguishing features were adopted by the Board for combination with the machine of Albert H. Emery, who obtained the contract, but have only been carried out sufficiently to show the correctness of the main principle. It is hoped that the whole apparatus may yet be constructed with the strength of parts, and in the original simple form proposed by the writer as devices for this purpose are all that are known to be lacking to perfect this remarkable machine.

theory. In this case the value of Y_0 was found equal to 5.202 $M \div b\ D^2$, the co-efficient being nearly 6 as by the ordinary theory. The final shifting of the neutral axis was trifling, the distance from same to outer fibre on the compressive side being .964 of that on the extended side. Yet this is one of the principal experiments made by Hodgkinson to prove his theory that the neutral axis shifts sufficiently to account for this discrepancy. It is expected, however, that further experiments will establish the fact that the curves near the origin have the same inclination for compression as for extension. The long bars experimented with (see § 12) may have buckled sufficiently to make part of the recorded compressions apparent rather than real before full bearing in guides was obtained.

§ 17. The transverse strength of the iron, expressed by a well known formula is $S = \frac{LW}{4bD^2} = 5580$ lbs. By the ordinary formula the strain on the outer fibre, representing the modulus of rupture of the material, would be six times the above, or 33480 lbs., and the consideration of the actual experimental curve in this case, reduces this to 5.202 S or 29027 lbs., whereas the actual ultimate tensile resistance by experiment was found to be only 15458 lbs., or less than one-half as much as shown by the ordinary theory, and more nearly half as much when the actual sequence of strains shown by the experimental curve is considered.

§ 18. By inspection of curve 2 showing the average results obtained by Mr. Hodgkinson from different kinds of iron, it will be seen that it cannot change the general conclusions materially, and as this represents the average of 16 experiments in compression and more in tension with different kinds of cast iron of apparently average quality, it becomes evident that for such material at least, other causes must be sought than the shifting of the neutral axis.

§ 19. For Curve 3, showing the results of experiments made by Capt. Rodman on a sample of gun iron, the formula gives $Y_0 = 3.2922\ M \div b\ D^2$. The value of S for transverse strain in formula in previous section was 5450, so in this case the ordinary formula would give for the ultimate strength of the material ($6 \times 5450 =$) 32 700 lbs., and when the experimental curve is considered, the modified formula gives for the ultimate strength ($3.292 \times 5450 =$) 17 941 lbs. The actual ultimate strength is stated to be 25 627 lbs., or less than given by the ordinary, and more than shown by the modified formula. In this case the neutral

axis shifted so that the depth of the compressed was but (.596 say) six-tenths of the extended section. There are several influences that may have modified the results. In the first place it is practically impossible to obtain the ultimate elongation by experiment, and we have obtained it from the approximate equation stated in § 54 Appendix, which gives the curve shown by the dotted line. Inspection will show, however, that probably no great error could occur in this way, the ultimate tensile resistance at *s*, on curve being fixed by experiment. Again, in making the experiments the strains were released after each additional thousand pounds, in order to ascertain the sets; and it is stated by Capt. Rodman himself, and other experimenters, that the resistance of iron is increased by giving the material an opportunity to rest between the several impositions of a load, the fibres appearing to rearrange themselves according to the new conditions;* so, probably the resistance given for the greater elongations are somewhat higher than would have been obtained had the time in which rupture was obtained been reduced to that probably occupied by the simpler transverse tests with which we have made comparisons.

§ 20. However, duly allowing for both these considerations, it is evident that even for material of this kind, the introduction of the experimental curve, thereby allowing for the shifting of the neutral axis in the formula, will not account for the discrepancy, the results being that, with ordinary cast iron with little extension, the amended formula still gives the ultimate tensile stress higher than it should be, as shown from Hodgkinsons' experiments; while, for gun iron, with very considerable elongation, such formula gives the ultimate stress lower than it should be.

§ 21. Curve 9, referring to Mr. Kirkaldy's tests of Swedish Bessemer steel, shows that the properties of the material are very different from those of cast iron. The curve within the elastic limit forms nearly a right line, in accordance with the ordinarily received opinion that the elongations are proportioned to the strains, and after the elastic limit is passed a nearly uniform, though slightly increasing, resistance is maintained, both in compression and tension, through a comparatively great range. It is interesting, too, to see how nearly alike the mechanical properties of the material are in tension in compression. The two elastic limits are nearly the same, though other experiments show this is affected somewhat by the hardness. For a material so closely fulfilling

* See Capt. Rodman's work, the Railway Commissioner's Report, and Prof. Thurston's papers, previously mentioned.

the requirements of construction and which are in great part assumed in theoretical calculation for all materials, it would naturally be supposed that the ordinary accepted theories of transverse resistance would give results agreeing, approximately at least, with those obtained from direct experiment, but enormous discrepancies are shown of the same nature as those found in the similar investigation with respect to cast iron.

§ 22. The accompanying table is an abstract from the experiments of Mr. Kirkaldy, previously mentioned—the abstract being more complete than necessary for this investigation, on account of the value of the results for reference.

1	Stamp on Specimen..................	1.2	0.9	0.6	0.3
2	Shearing strain per sq. in. Ultimate (Shearing edges hardened)..........	lbs. 61 412	79 737	71 648	45 410
3	Thrusting strain per sq. in. Elastic..	62 333	58 666	53 333	41 000
4	Do. Ultimate.	133 333	117 560	105 333	81 760
5	Effects of latter..................... (Specimens 4 diameters in length.)	Skewed	Skewed	Skewed	Buckled
6	Pulling stress per sq. inch. Elastic..	62 033	63 066	58 100	43 100
7	Do. Ultimate..	85 200	106 613	102 632	61 312
	Effects of latter.....................	Fractured.	Fractured	Fractured	Fractured
8	Transverse stress. Value of $S = \frac{LW}{4bD^2}$..	24 000	31 950	27 840	18 430
9	Corresponding strains on outer fibre by ordinary formula............. ...	144 000	191 700	167 040	110 980
10	Effects when experiment was stopped.	Fractured.	Fractured	Uncracked	Uncracked

These experiments were made with specimens of bars similarly hammered to two inches square. They were of different degrees of hardness—those at the left being the hardest. Shorter specimens gave higher resistances to compression, and longer ones lower resistances than those stated. Each column of results are the averages from three experiments.

§ 23. For this investigation, we have but to compare lines 7 and 9, which should agree, at least approximately, whereas it is seen that even with this superior material, the ultimate tensile resistance, calculated by the usual formula from the transverse resistance, gives results 70 and 80 per cent. in excess of the actual ultimate tensile resistances for the two harder grades of steel that were broken, and this discrepancy had reached

60 and 80 per cent. respectively, for the two specimens removed before breaking and would have been much greater, evidently, if such experiments could have been continued to rupture. It is stated that the bars of the last series were so soft as to readily permit bending in the form of a horseshoe. The limits of elasticity in tension and compression are so nearly alike,—sometimes the tensile, at others the compressive limit, being a little the higher, that the discrepancy cannot be in general attributed to the shifting of the neutral axis, although Mr. Kirkaldy, by noting the change of shape of circles on the bars, concluded that the axis by no means remained in the center of the bar. Evidently the ordinary theory does not apply either to cast iron or steel even when the latter is soft, and reasons other than those previously stated must be found to account for the discrepancy.

§ 24. During the progress of this investigation it occurred to the writer that an explanation might be suggested by ascertaining the isodynamic curves (or curves of equal strain) in accordance with theory, and plotting them in the same relative positions that they would occur longitudinally in a strained beam. Since a curve from an equation of the form $y = ax^n$ can be passed through any three points of any curve, for simplicity this form has been adopted in determining the equation of the isodynamic curve in Appendix B, § 57, and it has been found to apply about as well to the tensile branch of curve 1 as the more elaborate equation previously used. This equation reduces evidently to the ordinary form when $n = 1$.

§ 25. It is shown in Appendix B, § 61, that the ordinates of the isodynamic curve in terms of the depth of the section on one side of the neutral axis are, for a function in the form above stated, inversely proportional to the *nth* root of the moment. Figure 2 (on the plate) is intended to represent the side of a half beam loaded with a weight W in the centre, and supported at the ends ; and upon the same are plotted a series of isodynamic curves determined from equation (60) Appendix B, on the basis that the exponent $n = .79$, which was found applicable for the experiments with Low Moor iron, as explained in § 57 Appendix. It is suggested as highly probable, in fact almost a necessary inference, that on account of the lateral adhesion of the particles the strains of greater intensity near the top and bottom, and at the center of the beam, are in part transferred to the lamina of less strain in the direction of the curve nearer the axis, so that the fibres crossing the central section of maximum

moment are under greater strain than their proportional distance from the axis, and the outer fibres thereby correspondingly relieved.

§ 26. The action may be illustrated as follows: If strain be applied to the center of opposite sides of a piece of rubber the entire central section of the rubber between the points of application of the force will be strained; the strain being greatest in a line joining the forces, and reduced on either side. If, now, forces of less intensity be applied either side of the first force, the effect will be to increase all the strains somewhat, but a portion of the strain from the original force will still be carried by parts of the section not directly in line therewith, and the fibres directly in line be thereby correspondingly relieved. The question at once arises, how room can be found for increased elongations and compressions necessary to produce increased strains, between the neutral axis and outer edge of a beam? This is evidently provided by the shape of the isodynamic curves which run from the intermediate spaces either side of the axis to intermediate distances in the length of the beam, so an action of the kind mentioned should simply cause the beam to bend more either side of the centre than theory demands, which furnishes the basis, with sufficient delicate apparatus, of experimentally testing the correctness of the hypothesis.

§ 27. In support of this theory, attention is called to the fact that the shorter central isodynamic curves closely resemble the wedge shaped piece excluded when the material parts in compression, showing a concentration of force along the general direction of the curves. Again, curiously it appears from Mr. Nickerson's experiments with polarized light, previously mentioned, that the lines of strain in glass resemble the isodynamic curves presented very closely, and it having been shown that steel, even when soft, presents the same discrepancies of theory with experiment, for transverse strains, as cast iron, it appears probable that it is a general phenomenon for most or all materials, which has not been pointed out from the want of experiments as carefully conducted as those to which reference is made. Lastly, for the surplus strength shown by the very soft specimens of steel which bent very much before breaking this theory in connection with the results of longitudinal tests will account very satisfactorily; for by inspection of curve 9 it becomes evident that after the elastic limit of the outer fibres and many of the intermediate fibres is passed, such fibres are still offering a nearly constant resistance due to the horizontal part of the curve,

while the equalization of strain can still go on as before. The effect of an equalization of strain between the outer and medial fibres is to give the real isodynamic curves greater inclination from the horizontal than those derived from direct longitudinal experiments. From this it appears that the increased strains due to the diagonal direction of the curves may become so important that the fibres will part at one side of the section of maximum moment as is frequently reported.

§ 28. In investigating the general subject, the writer has not thought it proper to pass lightly over this problem which has remained unsolved for forty years or more since attention was first called to it, and although another theory is the final result, evidently a plausible hypothesis must always precede and point out the direction of experiments; and in any event the amount of information necessarily collected to investigate the matter will, it is hoped, interest others in the subject. Moreover, the investigation leads, it will be shown, to formula accurate in form and including probably all the principles involved, but which, if proved to be empirical by the further experiments suggested, will yet serve all practical requirements.

§ 29. As it has been thought better to separate the mathematical part of this paper in an appendix, a brief recapitulation may be of interest.

For a beam subject to transverse strain the moments of resistance of fibres at different distances from the neutral axis are generally calculated by a method due to Navier, based on the suppositions—1st, that the strains are proportioned to the elongations; and, 2d, that the elongations are proportioned to the distances of the fibres from the neutral axis. Thus if the distances last mentioned be represented by the variable x, this quantity finally becomes involved three times in the resulting expression, viz., once to represent the depth of the section (being advanced during integration from its differential, representing the depth of a fibre); once to represent the elongations, or the corresponding strains or unit forces due thereto, and again to give the moment of resistance of each fibre from the neutral axis; such moment assisting in balancing the external moment. It being known that the first of the principal suppositions is not in general true,—that is, that the strains are not proportioned to the elongations, at least for all values to the point of rupture,—it has been suggested that if the true relations between the same were included in the discussion, the well known discrepancies would disappear. Such relations have been included in the formula in this paper, by sub-

stituting for one of the factors, x, a function of x, showing the relations between the elongations and corresponding strains as determined by experiments (see § 15 and § 47), but it is found that by this means the discrepancies are only partially accounted for (§ 16, §§ 53–55).

The writer claims in this paper that the second supposition upon which the formula of Navier is founded, viz., that the elongations of the fibres are proportioned to their respective distances from the neutral axis is also incorrect, and that when the lateral adhesion of the fibres is considered, the positions of the isodynamic curves indicate both the possibility and probability that the intermediate fibres between the axis and outer boundaries of the beam may, and do receive more strain than that due to their positions, thereby relieving the outer fibres (§§ 24–27). This phenomenon can only account for somewhat less than 50 per cent. of the excess of resistance found by experiment (see § 38), and when the excess is greater than this, it is evidently due to the shifting of the neutral axis, which is provided for in the first branch of the inquiry.

§ 30. Equations (51) and (52) Appendix show that, both by the ordinary method and when the shifting of the neutral axis is considered, the strain on the outer fibre is expressed by an equation in the form

(61.) $Y = \frac{KM}{bD^2}$, K as shown in §§ 52–55, § 16 and § 19, reducing to a numerical co-efficient. The portion of the second term $\frac{M}{bD^2}$ will be recognized as equal to S in the well known formula for transverse strain,

(62.) $S = \frac{LW}{4\,bD^2}$ so from (61) we have

(63.) $Y = KS$.

When the function of x representing the relation of the strains to the elongations is in the form (53) $y = \varphi\,(x) = \alpha\,x^n$, and the neutral axis is considered to be in the centre, by combining (56), (20) and (62) we may have

(64.) $Y = 2\,(n + 2)\,\frac{M}{bD^2} = 2\,(n + 2)\,S$, so when Y and S are known, a value of α may be obtained, which in an equation of the form of (53) will give the actual relative strains on each fibre, with the equalization of strains between different fibres duly considered.

With materials for which the ultimate resistance is greater for compression than for tension, a modification of eq. (51) may be used to provide for the shifting of the neutral axis.

§ 31. In practice Eq. (63) may be treated as an empirical formula in which by ordinary theory, as shown by § 52, K equals 6, but when the other conditions herein mentioned are considered, is reduced to from 2.8 to 3.0 for ordinary cast iron; to about 4.7 for gun iron, and to 3.3 to 3.6 for moderately low steel from the hammer—not annealed or tempered. It is probable that the value for wrought iron is about the same as the latter. These values of K may be made more definite if members will be kind enough to report the results of experiments made by them on both the ultimate longitudinal and transverse resistances of materials of stated kinds and grades.

§ 32. Evidently for solid beams the equalization of strain referred to must commence as soon as flexure begins, which explains the fact that the ordinary formulæ are incorrect, even within the limits of elasticity. When experiments can be directed in the special direction desired, the formula for deflection may be put in a form similar to those developed for rupture in Appendix B. Considerable might be done in that direction with facts available, but will not now be attempted as the formula in Appendix A, based on the ordinary theory (see § 9), can be modified to answer all purposes in relation to skeleton girders which are more generally applicable in large structures.

APPENDIX A.

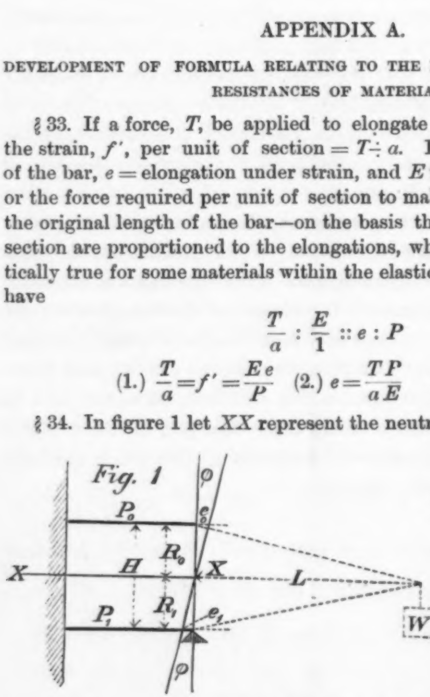
DEVELOPMENT OF FORMULA RELATING TO THE FLEXURE AND TRANSVERSE RESISTANCES OF MATERIALS.

§ 33. If a force, T, be applied to elongate a bar with a section, a, the strain, f', per unit of section $= T \div a$. If $P =$ the original length of the bar, $e =$ elongation under strain, and E the modulus of elasticity, or the force required per unit of section to make the elongation equal to the original length of the bar—on the basis that the strains per unit of section are proportioned to the elongations, which has been found practically true for some materials within the elastic limit by experiment, we have

$$\frac{T}{a} : \frac{E}{1} :: e : P$$

$$(1.)\ \frac{T}{a} = f' = \frac{Ee}{P} \quad (2.)\ e = \frac{TP}{aE} \quad (3.)\ T = \frac{aeE}{P}$$

§ 34. In figure 1 let XX represent the neutral axis of a girder. Distinguish symbols referring respectively to quantities above and below the neutral axis by the indices sub. $_0$ and sub. $_1$, and general quantities applicable to either or both sides of the axis by the same symbols without an index. Let $P_0\ P_1$ represent the full lengths or other portions of the chords of a skeleton girder subjected to longitudinal strains in resisting an external moment (for instance $LW =$) M. Let R'^0 and R'_1 equal the distances from the neutral axis to the centers of the chords; a'_0 and a'_1 the areas of the chords; E_0 and E_1 the moduli of elasticity, and e_0 and e_1 the elongations of the chords; N the horizontal strain on chords, and H the effective height of girder, or the distance to the center of the chords when connected by joints; then



$$(4.)\ H = R'_0 + R'_1 \quad (5.)\ N = \frac{M}{R'_0 + R'_2} = \frac{M}{H}$$

As one chord is lengthened and the other shortened a vertical section (at the right, per sketch) will move through a small angle, which represent by φ; the elongation of the chords will then be $\varphi R'_0$ and $\varphi R'_1$; and from (2) we have

$$(6.)\ e_0 = \varphi R'_0 = \frac{N P_0}{a'_0 E_0} \quad (7.)\ e_1 = \varphi R'_1 = \frac{N P_1}{a'_1 E_1}$$

Combining both equations with (5):

$$(8.)\ M = \frac{E_0 \varphi}{P_0} a'_0 R'_0 (R'_0 + R'_1) = \frac{E_1 \varphi}{P_1} a'_1 R'_1 (R'_0 + R'_1)$$

§ 35. If f = the force per unit of area or the unit force acting at the distance c from the neutral axis the extension due to this force $= c\,\varphi$. Substituting this as the value of e in (1) we have

$$(9.)\ \frac{f}{c} = \frac{E \varphi}{P}$$

When the chords are considered bent at every section the value of P above becomes dx, the distance between consecutive sections, and as φ and P are proportional, in such case we should substitute $\frac{E\,d\varphi}{dx}$ for $\frac{E\varphi}{P}$ As the angle φ is small it may be put equal to its tangent $\frac{dy}{dx}$ so $\frac{E\,d\varphi}{dx} = E\frac{d^2y}{dx^2}$. We may then use interchangeably, in all values of the external moment M, a factor:

$$\begin{array}{ccccccc} & (10) & & (11) & & (12) & & (13) \\ G = & \frac{f}{c} & \text{or} & \frac{E\varphi}{P} & \text{or} & \frac{E\,d\varphi}{dx} & \text{or} & E\frac{d^2y}{dx^2} \end{array}$$

according to the conditions of the problem or the purposes to which the equations are to be applied.

§ 36. Equation (8) may then be put in the following form :

(14.) $M = G_0 a'_0 R'_0 (R'_0 + R'_1) = G_1 a'_1 R'_1 (R'_0 + R'_1)$, and when the length sections and elasticity of the chords are equal, $R'_0 = R'_1$, and putting $2a' = A'$.

$$(15.)\quad M = G\,(2a'R'^2) = GA'R'^2 = G\frac{a'\,H^2}{2}$$

If u = the elongation of a chord we may have first, from (2), for either chord, and second, from (2) and (5), when the values of E and R' for both chords are considered equal,

$$(15a)\ u = \frac{N P}{a'\,E} \qquad (15b)\ u = \frac{M P}{a'\,E H}$$

$$(15c)\ du = \frac{N}{a'\,E} dx \qquad (15b)\ du = \frac{M}{a'\,E H} dx = \frac{M}{A'\,E\,R'} dx$$

$$(15e)\ M = a'\,E H \frac{du}{dx} = A' E R' \frac{du}{dx}$$

From § 34, $u = \varphi R'$, which substituted in (15e) with value of G from (12), gives (15), which, as will be seen hereafter, is equivalent to (23).

§ 37. In applying transverse strain to a solid beam, it is evident that the portions of the section above and below the neutral axis perform the same offices as the chords in the previous discussion. Conceive the

beam made up of an infinite number of longitudinal fibres, and let b represent the breadth and dv the thickness of a fibre, at a distance v from the neutral axis. Under the influence of the moment, consecutive vertical sections, parallel before flexure of the beam. will, by extension and compression of the fibres, move through a slight angle, which put equal $d\varphi$. Maxing $x =$ original length of the neutral axis, when the deflection is small, the projected length will remain practically the same; and as flexure takes place at every section, the length considered $= dx$, the distance between consecutive sections. The elongation of a fibre at a distance v from the neutral axis equals then $vd\varphi$. The force due to this elongation may be found from (3), and is an increment of the total force due to the change of length of all the fibres on one side of the neutral axis; hence,

(16.) $N = E\dfrac{d\varphi}{dx}\displaystyle\int b\,v\,dv$, so putting $h_0\ h_1$, equal to the distances of the outer fibres from the neutral axis, and using coefficient G from (12), we have,

$$(17.)\quad N = G_0\int_0^{h_0} b_0\,v\,dv = G_1\int_{h_1}^{0} b_1\,v\,dv = G_0\,\frac{b_0\,h_0^{\,2}}{2} = G_1\,\frac{b_1\,h_1^{\,2}}{2}$$

the latter values applying only when b_0 and b_1 are respectively constant.

§ 38. Putting r_0 and $r_1 =$ *radii of effect*, or distances from neutral axis to the centres of effort of all the forces for the extended and compressed sections respectively, evidently the moment of the total force N into these *radii* respectively equals the sums of the moments of the forces due to the changes in length of the several fibres on the corresponding sides of the neutral axis into their several distances from the axis. Hence multiplying each of the above forces by v, we have,

$$(18.)\quad Nr_0 = G_0\int_0^{h_0} b_0\,v^2\,dv = G_0\,b_0\,\frac{h_0^{\,3}}{3}$$

$$(19.)\quad Nr_1 = G_1\int_{h}^{0_1} b_1\,v^2\,dv = G_1\,b_1\,\frac{h_1^{\,3}}{3}$$

the latter values applying as before, only when b_0 and b_1 are respectively constant. The integrals respectively represent the moments of inertia of the extended and compressed sections, and will be distinguished by the symbols I_0 and I_1.

[Note referred to in § 29.] If by any means the resisting unit forces could be made the same at all distances from the axis one value of v, above, would be constant, making the denominators in second terms 2 instead of 3, so the aggregate resistances would be increased in the proportion of $\frac{1}{3}$ to $\frac{1}{2}$, or 50 per cent.

§ 39. The effective height H of a solid girder is $r_0 + r_1$, the distance between the centres of effort of the tensile and compressive forces; hence,

(20.) $M = N(r_0 + r_1)$ and adding equations (18) and (19), using value of G from (10) we obtain the somewhat more familiar form.*

$$(21.)\ M = \frac{f_0}{c}\int_o^{h_0} b_0\, v^2\, dv + \frac{f_1}{c}\int_{h_1}^{o} b_1\, v^2\, dv.$$

§ 40. On the basis that the moduli of elasticity are alike in tension and compression, taking value of G from (12) and integrating together the expressions for the upper and lower sections another form is obtained viz: †

$$(22.)\ M = \frac{E\, d\varphi}{dx}\int_{-h = h_1}^{+h = h_0} bv^2\, dv\,.$$

The integral in this case is the moment of inertia of the entire section of the beam, and is designated I. Substituting the value of G from (13) in the last equation we have

(23.) $M = E\, I \dfrac{d^2y}{dx^2}$ which is the well known equation of the elastic line. Integrating once gives the tangent of the angle of curvature due to the deflection, and a second integration the deflection itself. The angle and deflection at any point of a girder may therefore always be de termined when an expression can be found for the moment.

§ 41. Equations (15), (21) and (22) show that the resistance offered by the extension of each of a series of bars or fibres at different distances from the neutral axis is proportioned to its section into the square of its distance from the axis. By dividing the moment of inertia I_0 of the extended section by the area a_0 of the latter, the quotient represents the square of the distance from the neutral axis at which that area must be concentrated to have the same moment of resistance as when distributed in the actual section. Let this distance, called the *radius of gyration* be represented by R_0, then

$$(24).\ \frac{I_0}{a_0} = R_0^2 \quad (25.)\ R_0 = \sqrt{\frac{I_0}{a_0}} \quad (26.)\ I_0 = a_0\, R_0^2$$

§ 42. The relations between the similar quantities for the compressed section will be expressed by the same equations with the index $_0$ changed to $_1$. So also when the upper and lower portions of the beam are symmetrical and the moduli of elasticity and rupture in tension and compression equal, by dividing the moment of inertia I of the *whole* section by the *total* area A, we have

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*** Compare Stoney on Strains, § 70. † Compare Du Bois Graphical Statics, page 112.**

(27.) $I = AR^2 = \frac{aH^2}{2}$ and combining this with (23) and (13) we may have

(28,) $M = G\,I = G\,A\,R^2 = G\frac{aH^2}{2}$ which corresponds in form with (15), showing that when the gross area of equal chords and of a beam of symmetrical section are equal, $R' = R$. Evidently also unsymmetrical beam sections may be put in the form of (14), so it appears that the moment of resistance of any beam is equal to that of two chords of a girder having respectively the same areas as the sections of the beam above and below the neutral axis, when the centers of the chords are arranged at distances from the axis equal to the respective radii of gyration of the sections of the beam.

§ 43. The *radius of gyration* should not be confounded with what we have above termed the *radius of effect*. The former is the distance from the axis at which a certain area uniformly elongated in all parts of its section will support a given moment—the latter the acting distance of the *forces* arising from unequal elongation of different parts of the same area. The radii of effect are then only to be considered in connection with unequal elongations, but the integral of the forces arising therefrom being a definite quantity may arbitrarily be put equal to other quantities which are made thereby of equal value. In (27), since A is the area of the total section, R is the radius of gyration, but it is evident from the corresponding, equation (15), that A' may be any area or R' any distance, so long as $A'\,R'^2 = I$.

From (25) and (27) we may have for b constant—

(29.) $$R = \sqrt{\frac{I_0}{a}} = \sqrt{\frac{b\,h^3}{3\,b\,h}} = .577\ h$$

and from (17) and (18)—

(30.) $$r = \frac{2}{3}h.$$

§ 44. That is the centre of effort of the forces due to unequal elongation of the fibres is at a greater distance from the axis than the radius of gyration, as it evidently should be. It is not necessary to find the value of r for symmetrical sections when the modulus of elasticity is considered constant, but it has an important bearing in the more general consideration of the subject, for which preparation has now been made.

§ 45. Resuming for a time the consideration of a skeleton girder we may obtain from (8) by equating the two values of M.

(31.) $a_0\,E_0\,R_0\,P_1 = a_1\,E_1\,R_1\,P_0$ which shows that the distances R_0 and R_1 and consequently the elongation of the chords which are proportional thereto, vary directly as the lengths elongated, and inversely as the areas and moduli of the chords. The lengths P_0 and P_1 elon-

gated will generally be equal. In such cases which include those where flexure takes place at every section we may have.

(32.) $R_0 : R_1 :: a_1 \, E_1 : a_0 \, E_0$.

Hence, when the modulus is constant, the neutral axis divides the effective height H into distances R_0 and R_1 inversely proportional to the adjacent areas of chords, or practically of the flanges of a beam, and when the areas of the chords are equal the distances R_0 and R_1 are inversely proportional to the moduli.

§ 46. From (14) with value of G from (10) we have—

(33.) $\frac{f_0}{c_0} a_0 \, R_0 = \frac{f_1}{c_1} a_1 \, R_1$. Making the values of c, equal those of R, and the values of f equal $T_0 \; T_1$ the moduli of rupture of the material in tension and compression respectively, we have—

(34.) $a_0 : a_1 :: f_1 : f_0 :: T_1 : T_0$:

That is the areas of the chords of a girder should be proportioned to the ultimate strengths or moduli of rupture of the material in tension and compression, which is evidently true (though not always practicable on account of crippling), and explains the efficiency of the Hodgkinson beam in connection with principles previously adverted to, § 7. It will be noticed however that this expression takes no account of the deflection previous to rupture.

APPENDIX B.

DEVELOPMENT OF FORMULA FOR TRANSVERSE STRAINS ON THE BASIS THAT THE RELATIONS OF THE STRAINS TO THE ELONGATIONS VARY ACCORDING TO EXPERIMENT, INSTEAD OF IN PROPORTION TO THE ELONGATIONS.

§ 47. Representing the abscissæ of the curves as plotted by x, and the ordinates by y, the value of the latter may be expressed.

(35.) $y = \varphi(x)$. Substituting x in place of v in (17) and (18), the value of the function of x may be used instead of one factor, x. The integral of (17) will then give the area of the plotted curve, between the limits assigned, and of (18) the moments of its ordinates, which may also be obtained graphically when necessary or more convenient. The values of these integrals must then be modified to represent the relations in terms of the actual values of the elongations and strains. Putting Y = actual strain where the corresponding ordinate of curve $= y$, and s = the maximum value of x considered, the special values due to the scale of the plotted curve may be eliminated by dividing the integral by a special value of y, which is involved once, and by the value of s, repeated as many times as x is involved after integration. When the special values are eliminated, s refers only to the depth of the section, and h is substituted

in its place. The equations referring to the extended section based on (17) and (18) are then—

(36.) $$N=\frac{Y_0\, b_0\, h_0}{ys}\int_0^s \varphi(x)dx$$

(37.) $$Nr_0=\frac{Y_0\, b_0\, h_0^2}{ys^2}\int_0^s \varphi(x)xdx$$

§ 48. Unless specifically mentioned y_0 will hereafter refer to the maximum tensile ordinate, when $x=s$ and y_1 to the compression ordinate, when $x=s$, but which will not be the maximum ordinate in case the neutral axis shifts. If the strains in the compressed section be considered proportional to the elongations, as most of the curves show is practically true within the limits required, and the tangent of the angle at the origin be put equal to g, then $y=gx$ and—

(38.) $y_1=gs$ and we may have

(39.) $$N=\frac{Y_1\, b_1\, h_1}{gs^2}\int_s^0 gxdx=\frac{b_1\, h_1\, Y_1}{2}$$

(40.) $$Nr=\frac{Y_1\, b_1\, h_1^2}{gs^3}\int_s^0 gx^2\, dx=\frac{b_1\, h_1^2\, Y_1}{3}$$

§ 49. The final results in this case are the same of course as would be obtained by using equations (17) and (19) direct. Putting the numerical co-efficients derived from the integrals in (36) and (39)$=m_0$ and m^1 respectively, and, those from (37) and (40)$=n_0$ and n_1 respectively, we have at once (41) $m_1=\frac{1}{2}$ (42) $n_1=\frac{1}{3}$. Put also $\frac{y_0}{y_1}=t$ and $\frac{b_0}{b_1}=j$

§ 50. Equating (36) and (39), we have

(43.) $m_0\, b_0\, h_0\, Y_0=m_1\, b_1\, h_1\, Y_1$, hence

(44.) $$Y_1=2m_0\frac{b_0\, h_0}{b_1\, h_1}Y_0$$

We may have also

(45.) $Y_0 : Y_1 :: y_0 : y_1\frac{h_1}{h_0}$ which permits the neutral axis to shift by modifying the value of Y_1 in proportion to the decrease of h_1 in relation to h_0, hence

(46.) $$Y_1=\frac{y_1\, h_1}{y_0\, h_0}Y_0=\frac{h_1}{th_0}Y_0$$

§ 51. Equating (44) and (46), we find, easily,

(47.) $$b_1\, h_1^2=2m_0\, t\, b_0\, h_0^2$$

(48.) $h_1=\sqrt{2m_0\, jt}\, h_0$, hence

(49.) $$D=h_0+h_1=(1+\sqrt{2m_0\, jt})\, h_0$$

From (20), (37) and (40) we have

(50.) $$M=N(r_0+r_1)=n_0\, b_0\, h_0^2\, Y_0+\frac{b_1 h_1^2\, Y_1}{3}$$
$$=\left(n_0+\frac{2m_0}{3}\sqrt{2m_0\, jt}\right)b_0\, h_0^2\, Y_0,$$

and when $b_0 = b_1$

$$(51.)\ Y_0 = \frac{M}{\left(n_0 + \frac{2m_0}{3}\sqrt{2m_0\,t}\right) b_0 h_0^2} = \frac{(1+\sqrt{2m_0\,t})^2 M}{\left(n_0 + \frac{2m_0}{3}\sqrt{2m_0\,t}\right) b_0 D^2}$$

§ 52. When the neutral axis is central, the breadth uniform and the strain referred to the outer fibre, we may have from (21), and in another form from (50).

(52.) $f = Y_0 = \frac{6M}{bD^2}$ which, according to the ordinary theory, should equal the ultimate longitudinal resistance of the material per unit of section, but which, as has been explained, always gives in respect to materials mentioned, results much in excess of those obtained by direct application of longitudinal strain.

§ 53. Equation (51) was first applied to one of the curves numbered 8, which being taken from a tortion machine, it was thought would from increased elongations show a maximum shifting of the neutral axis. An equation in the form $y = ax^3 - bx^2$ was found for positive fractional values of x, to give the upper portion closely within the limits desired, and the compression and one-half the tension branch was considered a straight line in accordance with popular views on the subject. On this basis Y_0 was found equal to 4.316 $M \div bD^2$, or insufficient to account for the discrepancy (compare with § 52).

§ 54. The upper part of curve 3 was found to be represented approximately by the equation $y = 28.977 - \frac{122.62}{x + 3.547}$ which gives the curve shown by the dotted line. A portion of the tensile and a larger portion of the compression curve is represented closely by the inclined straight line shown, the equation of which is $y = 3.482$. This line being tangent to the tensile curve where $\frac{dy}{dx} = 3.482$ and $x = 2.387$. The results are given in § 19.

§ 55. Curve 7, which, as stated previously, was obtained from experiments on direct tension, was found to agree closely with the equation $y = 36.75 - \frac{202.125}{x + 5.5}$ and on the basis that the line representing the co-efficient of resistance was tangent to the curve of elongation at the origin, Y_0 was found equal to 4,061 $M \div bD^2$, in which the co-efficient is considerably too large to account for the discrepancy.

§ 56. No calculations have been made in respect to curves 5 and 6. They are presented as part of the evidence on the subject, but not agreeing with the other curves in the angle of the compressive branch compared with the tensile, are reserved for further investigation.

§ 57. For convenience in laying down the isodynamic curves referred to in § 24–25, the approximate equation of some of the curves was found in the simple, general form of

(53.) $y = \varphi(x) = ax^n$ The equation

(54.) $y = 1.5305\, x^{.79}$ represents very satisfactorily the tensile branch of curve 1, showing the results of Mr. Hodgkinson's experiments with Low Moor Iron No. 2, and an equation of this form can always be applied to such portions of any curve as are most important (see § 24). The function in this form is convenient and useful, from the fact that a simple solution may be obtained before substituting in any special values.

§ 58. From an inspection of (18) and (37) and explanations in § 23 it is evident that the moment of resistance of all the fibres in one section, that extended, for instance, is expressed by the equations referred to, entirely independent of what the other section may be. Nr, which put equal m', equaling a certain moment, independent of the value of the total moment, $M = N\ (r_0 + r_1)$. Substituting the value of $\varphi(x)$ from (53) in (37), we have then

$$(55.)\quad Nr = m' = \frac{Y_0\, b_0\, h_0^{\,2}}{y\, s^2} \int_0^s ax^{n+1}\, dx = \frac{Y_0\, b_0\, h_0^{\,2}}{as^{n+2}} \left(\frac{as^{n+2}}{n+2}\right) = \frac{Y_0 b_0\, h_0^{\,2}}{n+2}$$

$$(56.)\quad Y_0 = \frac{m'\,(n+2)}{b_0\, h_0^{\,2}}$$

§ 59. If Y represent the strain on any fibre at a distance x from the axis, Y_1 representing the strain on outer fibre when $x = s$ we may have from (53).

(57) $Y_0 : Y :: as^n : ax^n$ from which proportion and (56) we may have

$$(58)\qquad Y = \frac{x^n}{s^n}\, Y_0 = \frac{m'\ (n+2)}{b_0\, h_0^2} \left(\frac{x}{s}\right)^n$$

§ 60. If we put $k = \frac{x}{s}$, or the proportion of s considered, it will, when s takes its ultimate value, h_0, § (47), represent the proportion of the depth of the tensile section considered, and we may have

$$(59)\qquad Y = \frac{k^n\, m'\ (n+2)}{b_0\, h_0^{\,2}} \qquad\qquad (60)\qquad k = \left(\frac{Y b_0\, h_0^2}{m'\ (n+2)}\right)^{\frac{1}{n}}$$

§ 61. From the above it will be seen that for any fibre parallel with the neutral axis (k constant) the strain or unit force Y is proportioned to the moment. For Y constant (60) is the equation of the isodynamic curve, and shows that k, or the proportional distance from the neutral axis in terms of the height, varies inversely as the nth root of the moment—all the terms in the equation being constant except m'. The isodynamic curves are discussed in connection with illustrations in § 25 of the paper.